Euler Tours on the grid

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Goal: to experimentally evaluate whether the Markov chain that uses Kotzig-transformations on grids may be rapidly mixing.

Motivation: it would indicate whether there might be an efficient approximation algorithm for the number of Euler Tours on grids.

Approach: sampled a large number of Euler Tours and selected a number of pairs; built paths of tours in an ordered fashion using the Kotzig-transformation; computing an estimate of the edge loading to infer whether the chain using this transformation might be rapidly mixing.

COUNTING EULER TOURS

To check whether a graph has an Euler Tour is not too difficult - it can be solved in linear time.

A more challenging problem is to count the number of Euler Tours in the case of Eulerian graphs. It was proved that this problem is $\sharp$P-complete, where $\sharp$P is the class of counting problems associated to decision problems in NP.

Finding a polynomial time algorithm for a $\sharp$P-complete problem would imply that NP = P. However, we could try to find a polynomial time algorithm that, using randomness, produces an estimate that is sufficiently close to the real answer.

Markov Chains

To build a polynomial time approximation algorithm, the Markov Chain Monte Carlo method has proved to be really useful.

The Markov chain is required to have some specific properties to insure that a reliable approximation can be found:

- ergodicity
- the correct stationary distribution
- rapid mixing time

Conductance and canonical paths

To decide if a Markov Chain is *rapidly mixing*, we study its conductance - informally, a large conductance means there is no blockage in the chain, which means it has a fast mixing time.

Calculating this quantity directly involves the whole state space and considering all its subsets. Instead, we have used a technique called *canonical paths*.

This option was explored by Tetali and Vempala on low height grids, but there were found some errors in their proof. We have tried a modified approach.

The Kotzig Chain

In order to transform one Euler Tour of a graph into a different one, all is needed is a finite set of simple local transformations - called κ-transformations.

The Kotzig Chain is defined on the set of Euler Tours of a graph with the transitions being κ-transformations:

1. Choose an arbitrary Euler Tour of the graph $G = (V, E)$.
2. From the set of vertices V pick one uniformly at random.
3. With probability $1/2$ make the only possible κ-transformation at v and with the remaining probability do nothing.

The Kotzig Chain

To study the conductance of this chain, we have measured the edge loading in paths between states of the state space using κ-transformations.

For this we first needed to sample a large number of Euler Tours of the grids.

Dynamic programming algorithm

To sample Euler Tours we have used a dynamic programming approach similar to Dyers algorithm for sampling knapsack solutions.

First step: build a table iteratively, adding the number of paths corresponding to some cycle decomposition of tours.

Second step: probabilistically backtrack on the table and build the tour.

This approach works because the counts for Euler Tours can be built up inductively on grids. When their height is low we can compute the number of tours directly using the *transfer matrix*.

THE TRANSFER MATRIX APPROACH

In grids, cylinders and toruses with fixed height, a collection of graph structures (e.g., spanning trees, Hamiltonian cycles, independent sets, acyclic orientations) should be partitionable in terms of the configuration on the left and right columns.

The counts of such structures can probably be built up inductively as columns are added. For an appropriate transfer matrix A and corresponding row vectors x and y the computation should be achievable as

$$|ET(G(m,n))| = \vec{x}A^n\vec{y}^T.$$

BUILDING THE PATHS

Given two tours A and B, the first step is to build the disagreement set - this is done by comparing the pairings of edges at each vertex in the tours, in order.

The next step is to iterate through this set, "fixing" the pairing of edges in the current tour so that it matches the pairing of edges of that vertex in the final tour. Depending on the availability of the κ-transformation, this step may involve visiting multiple states of the state space. Every time a state is visited, the count of the corresponding edge is increased.

Once we have built a path, another pair of tours is generated and the above algorithm repeated.

ANALYSIS

We experimentally evaluated conductance over paths between randomly generated pairs of Euler Tours.

For a chain to be rapidly mixing, it needs to have high conductance, so the collection of paths should not have a high edge loading.

An estimate of the edge loading is given by:

$$\rho_{estimate}(\Gamma) = \frac{n(|\Omega| - 1)}{P} \max_e \sum_{\gamma_{xy} \ni e} |\gamma_{xy}|.$$

ANALYSIS

We have run the path building algorithm on grids of sizes $(2, n)$ and $(3, n)$ with n between 5 and 10. We have sampled 10^6 pairs for the $(2, n)$ grids and 10^5 pairs for the $(3, n)$ grids.

$(2, n)$	5	6	7	8	9	10
tours	$1.6 \cdot 10^4$	$1.1 \cdot 10^5$	$7.9 \cdot 10^5$	$5.3 \cdot 10^6$	$3.5 \cdot 10^7$	$2.3 \cdot 10^8$
$\rho_{estimate}$	$2.5 \cdot 10^2$	$5.5 \cdot 10^2$	$1.5 \cdot 10^3$	$7.1 \cdot 10^3$	$3.7 \cdot 10^4$	$2.2 \cdot 10^5$

$(3, n)$	5	6	7	8	9	10
tours	$1.7 \cdot 10^6$	$2.8 \cdot 10^7$	$4.5 \cdot 10^8$	$7.2 \cdot 10^9$	$1.1 \cdot 10^{11}$	$1.7 \cdot 10^{12}$
$\rho_{estimate}$	$6 \cdot 10^3$	$1.4 \cdot 10^5$	$2.4 \cdot 10^6$	$4.6 \cdot 10^7$	$6.8 \cdot 10^8$	$1.3 \cdot 10^{10}$

The project aimed to experimentally study whether the Kotzig Chain may be rapidly mixing, which would indicate that an efficient approximation algorithm for the number of Euler Tours on graphs might be found.

For this there were three main objectives:

- sampling Euler Tours on graphs,
- building the paths between pairs of tours,
- analyse the conductance based on the experimental results.